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Soliton mass and discrete coupling constants in the quantized sine-Gordon model

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Abstract. The soliton mass in the quantized sine-Gordon model over the whole range of the coupling constant (g^2 from 0 to 8π) is evaluated by means of the Gaussian effective potential method. The analytical expression with numerical results is presented. We find that the coupling constant g can only take discrete values. A new kind of soliton linking $g\Phi = 0$ to $g\Phi = \pi$ emerges at a set of special coupling constants g.

1. Introduction

The sine-Gordon (s-G) model in $(1+1)$ dimensions is an ideal theoretical laboratory for nonlinear phenomena in physics, either in the classical or the quantum version. For an excellent review, see e.g. [1]. Recently, the method of Gaussian effective potential (GEP) in quantum field theory has been applied to the s-G model with impressive results [2-5]. While in that literature the quantization of the s-G model is carried out in uniform configuration, we will concentrate in this paper on the quantization of the s-G model in a non-uniform background, i.e. the quantization around a soliton. In other words, we will discuss the quantum correction of a soliton mass by the GEP method and compare our result with that of [6] and [7].

The organization of this paper is as follows. After a brief description of the GEP method in section 2, we shall improve it for the case of uniform background in section 3 to derive the mass of the quantized soliton. Some analytic and numerical results will be given in section 4. Section 5 contains a brief summary and discussion. The zero mode and related problems are discussed in an appendix.

2. *The Gaussian effective potential method* [2-5, 7]

We begin with the Lagrangian density of the s-G model

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^4}{\lambda}\left[\cos\frac{\sqrt{\lambda}}{m}\phi - 1\right] \tag{2.1}$$

and write down the Hamiltonian of this system as

$$H = \int \mathrm{d}x\left[\tfrac{1}{2}\Pi_\phi^2 + \tfrac{1}{2}(\partial_x\phi)^2 - \frac{m^4}{\lambda}\left(\cos\frac{\sqrt{\lambda}}{m}\phi - 1\right)\right] \tag{2.2}$$

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($\Pi_\phi = \partial\mathscr{L}/\partial\dot{\phi} = \dot{\phi}$). This GEP method amounts to introducing a Gaussian wavefunctional (GWF) with an external source $J(x)$:

$$|\bar{\Psi}\rangle_J = N_f \exp\left\{\frac{i}{\hbar}\int dx\, \mathscr{P}_x\phi_x - \frac{1}{2\hbar}\int dx\, dy(\phi_x - \Phi_x)f_{xy}(\phi_y - \Phi_y) + \tfrac{1}{2}\int dx\, J_x\phi_x\right\} \quad (2.3)$$

where $\phi_x = \phi(x)$, $\Phi_x = \langle\Psi|\phi_x|\Psi\rangle_{J=0}$. The normalization condition $\langle\Psi|\Psi\rangle_{J=0} = 1$ leads to

$$\langle\Psi|\Psi\rangle_J = \exp\left\{\int dx\, J_x\Phi_x + \tfrac{1}{4}\hbar \int dx\, dy\, J_x f^{-1}_{xy} J_y\right\} \quad (2.4)$$

where $f^{-1}_{xy} = f^{-1}(x-y)$ is the inverse of the quantum fluctuation correlation function $f_{xy} = f(x-y)$ such that

$$\int dy\, f(x-y)f^{-1}(y-z) = \delta(x-z) \quad (2.5)$$

with their Fourier transformations ($\hbar = 1$):

$$f_{xy} = \int \frac{dp}{2\pi} \exp[ip(x-y)] f_p \quad (2.6)$$

$$f^{-1}_{yz} = \int \frac{dk}{2\pi} \exp[ik(y-z)] \frac{1}{f_k}. \quad (2.7)$$

Evaluating the total energy of the s-G system in the state $|\Psi\rangle_{J=0}$ and using the trick explained in appendix A of [4], one obtains

$$\begin{aligned} E[\Phi, \mathscr{P}, f] = \langle\Psi|H|\Psi\rangle_{J=0} &= \int dx\, \varepsilon \\ &= \int dx \left\{\frac{1}{2}\left[\mathscr{P}^2_x + \frac{1}{2}f_{xx}\right] + \frac{1}{2}\left[(\partial_x\Phi_x)^2 - \frac{1}{2}\int dy\, \delta(x-y)\partial^2_x f^{-1}_{xy}\right]\right. \\ &\quad \left. + \frac{m^4}{\lambda}\left[1 - Z_\Omega \cos\frac{\sqrt{\lambda}}{m}\Phi_x\right]\right\} \end{aligned} \quad (2.8)$$

where

$$Z_\Omega = \exp\left[-\frac{\lambda}{4m^2} f^{-1}_{xx}\right]. \quad (2.9)$$

The variation of E with respect to f_p

$$\frac{\delta E}{\delta f_p} = 0 \quad (2.10)$$

yields

$$f_p = \sqrt{p^2 + \Omega^2} \quad (2.11)$$

with

$$\Omega^2 = m^2 Z_\Omega \cos\frac{\sqrt{\lambda}}{m}\Phi_x. \quad (2.12)$$

Moreover, $\delta E/\delta\mathcal{P}_x = 0$ leads to $\mathcal{P}_x = 0$. If considering the uniform configuration $\Phi = \text{constant}$, $\partial_x\Phi = 0$, one may define a GEP:

$$V_{\text{eff}}(\Phi) = \min_{\Omega^2} \varepsilon(\Phi, \Omega^2) = \varepsilon(\Phi, \mu^2(\Phi)) \tag{2.13}$$

via

$$\left.\frac{\partial\varepsilon}{\partial(\Omega^2)}\right|_{\Omega^2=\mu^2(\Phi)} = 0. \tag{2.14}$$

Hence

$$\mu^2(\Phi) = m^2 Z_\mu \cos\frac{\sqrt{\lambda}}{m}\Phi. \tag{2.15}$$

After performing a renormalization of parameters such that

$$m_{\text{R}}^2 = \left.\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\Phi^2}\right|_{\Phi=0} \tag{2.16}$$

$$\lambda_{\text{R}} = \left.\frac{\mathrm{d}^4 V_{\text{eH}}}{\mathrm{d}\Phi^4}\right|_{\Phi=0} \tag{2.17}$$

we find

$$m_{\text{R}}^2 = m^2 Z_\mu|_{\Phi=0} \tag{2.18}$$

$$g_{\text{R}}^2 \equiv \frac{\lambda_{\text{R}}}{m_{\text{R}}^2} = Z_g g^2 \equiv Z_g\frac{\lambda}{m^2} \tag{2.19}$$

$$Z_g = \frac{2g^2+8\pi}{8\pi-g^2}. \tag{2.20}$$

The mass parameter $\mu^2(\Phi)$ is related to m_{R}^2:

$$\mu^2(\Phi) = m_{\text{R}}^2(\cos g\Phi)^{8\pi/(8\pi-g^2)} \tag{2.21}$$

while the effective potential reads

$$V_{\text{eff}}(\Phi) = \text{constant} + \frac{m_{\text{R}}^2}{g^2}\left(\frac{g^2}{8\pi} - 1\right)(\cos g\Phi)^{8\pi/(8\pi-g^2)}. \tag{2.22}$$

Clearly, when $g^2 < 8\pi$, $\Phi = 0$ corresponds to the stable phase, while the stability criterion for the S-G model at the quantum level implies that

$$\left.\frac{\partial^2\varepsilon}{\partial(\Omega^2)^2}\right|_{\Omega^2=\mu^2(\Phi),\Phi=0} = \frac{1}{8\pi m_{\text{R}}^2}\left(1 - \frac{g^2}{8\pi}\right) > 0. \tag{2.23}$$

So one rediscovers the famous Coleman critical value of g^2, $g_{\text{cr}}^2 = 8\pi$ [8].

3. The expression of the soliton mass in the GEP method

Now we are facing a new problem. Can we still use the GEP method described in section 2 to calculate the mass of the quantized soliton in the S-G model? Notice that in the renormalization scheme used to derive the effective potential (2.22), we have distinguished $\Phi \neq 0$ from $\Phi = 0$. If Φ takes a uniform configuration, it is trivial. But it would be more meaningful for Φ to carry an x-dependence. Then the mass parameter Ω^2 in equation (2.12) is endowed with an x-dependence, which in turn will bring x-dependence to f_p in (2.11). Is it reasonable or not? Actually, if there is a non-uniform background, the quantum fluctuation correlation function f_{xy} should not have the property of translational invariance. So if we still expand it in terms of equation (2.6), the function f_p should in general carry a dependence on variable $(x+y)$. Since there is always a function $\delta(x-y)$ in front of f_{xy} or f_{xy}^{-1}, we eventually find

$$f_p(x)=\sqrt{p^2+\mu^2(x)}=\sqrt{p^2+m^2 Z_\mu \cos g\Phi(x)} \tag{3.1}$$

which should be viewed as

$$f_p(x+y)=f_p=\sqrt{p^2+m^2 Z_\mu \cos g\Phi\left(\frac{x+y}{2}\right)} \tag{3.2}$$

such that the symmetry $f_{xy}=f_{yx}$ is maintained. Then some remarks are in order:

(i) The term $\frac{1}{4}\int \mathrm{d}y\, \delta(x-y)\partial_x^2 f_{xy}^{-1}$ in the energy density will acquire an extra contribution

$$-\frac{1}{8\pi}\partial_x\left[\frac{1}{\mu^2}\frac{\partial\mu^2}{\partial\Phi}\partial_x\Phi\right] \qquad \left(\frac{\partial\mu^2}{\partial\Phi}=\frac{m^2 Z_\mu g \sin g\Phi}{g^{2/8\pi}-1}\right). \tag{3.3}$$

Since it turns out to be a total derivative, it has no influence on the equation for $\Phi(x)$. Of course, there is also no change in the equation for f_p.

(ii) Once when $\cos g\Phi(x)<0$, the mass parameter $\mu(x)$ becomes imaginary. One needs to check all the calculations up to the effective potential $V_{\mathrm{eff}}(\Phi)$. Fortunately, every formula remains valid irrespective of the sign of μ^2. (We confine ourselves to the real value of μ^2, see below.)

(iii) Last, but not least, we need to see whether the fundamental condition for f_{xy} and f_{yz}^{-1} remains valid $(\mu^2(x+y)=m^2 Z_\mu \cos g\Phi[(x+y)/2])$.

$$\int \mathrm{d}y \int \frac{\mathrm{d}p}{2\pi}\exp[\mathrm{i}p(x-y)][p^2+\mu^2(x+y)]^{1/2}\int\frac{\mathrm{d}k}{2\pi}\exp[\mathrm{i}k(y-z)][p^2+\mu^2(y+z)]^{-1/2}$$
$$\simeq \delta(x-z). \tag{3.4}$$

Notice that the well known Fourier transformation

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\frac{\exp[\mathrm{i}p(x-y)]}{\sqrt{p^2+\mu^2}}=\frac{1}{\pi}K_0(\mu|x-y|) \tag{3.5}$$

remains valid even when $\mu(x+y)$ is imaginary because

$$\begin{aligned}K_0(z)&=-\frac{\pi}{2}\mathrm{i}\,H_0^{(2)}(-\mathrm{i}z)\\ H_0^{(2)}(z)&=J_0(z)-\mathrm{i}N_0(z)\end{aligned} \tag{3.6}$$

with J_0 and N_0 being Bessel and Neumann functions of zero order. Define

$$\begin{aligned}G(\xi)&\equiv G(\mu|x-y|)=\int\frac{\mathrm{d}p}{2\pi}\sqrt{p^2+\mu^2}\exp[\mathrm{i}p(x-y)]\\ \frac{\partial G}{\partial\mu^2}&=\frac{1}{2\pi}K_0(\mu|x-y|)\\ \frac{\partial G}{\partial(|x-y|)}&=-\frac{\mu^2}{\pi|x-y|}K_0(\mu|x-y|).\end{aligned} \tag{3.7}$$

For $\mu^2 = \text{constant} > 0$, we have from (3.4)

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}y\, G(\mu|x-y|)K_0(\mu|y-z|) = \delta(x-z) \tag{3.8}$$

which could be viewed as a new expression for the δ function.

One may understand equation (3.8) by plotting the product of two functions G and K in the integrand. Now μ changes with y and in some region the argument of $G(\mu(x+y)|x-y|)$ and $k(\mu(y+z)|y-z|)$ may become imaginary. In this case we can still expect equation (3.8) to hold approximately.

The accuracy of this approximation could be seen alternatively as follows. Denote $f_p = \tilde{f}(p, R) = \sqrt{p^2+\mu^2(R)}$, then we have

$$\tilde{f}\left(p, \frac{x+y}{2}\right) = \tilde{f}\left(p, \frac{y+z}{2}\right) + \tilde{f}'\left(p, \frac{y+z}{2}\right)\frac{(x-z)}{2} + \dots$$

where

$$\tilde{f}'\left(p, \frac{y+z}{2}\right) = \left.\frac{\partial \tilde{f}(p,R)}{\partial R}\right|_{R=(y+z)/2}.$$

Hence

$$\begin{aligned}\int \mathrm{d}y f(x,y) f^{-1}(y,z)&\\ &= \int \mathrm{d}y \int \frac{\mathrm{d}p}{2\pi}\int \frac{\mathrm{d}k}{2\pi}\frac{\tilde{f}(p,(y+z)/2)}{\tilde{f}(k,(y+z)/2)}\\ &\quad\times\left\{1+\frac{\tilde{f}'(p,(y+z)/2)}{\tilde{f}(p,(y+z)/2)}\left(\frac{x-z}{2}\right)+\dots\right\}\mathrm{e}^{\mathrm{i}(px-kz)}\,\mathrm{e}^{\mathrm{i}y(k-p)}.\end{aligned}$$

Since $\tilde{f}(p,R)$ is a slowly varying function of R, we may substitute the ratio $\tilde{f}(p,R)/\tilde{f}(k,R)$ by its average with respect to $R[=(y+z)/2]$, $\overline{[\tilde{f}(p,R)/\tilde{f}(k,R)]}$, which is then independent of y when performing the integration of y. The ratio $\tilde{f}'(p,R)/\tilde{f}(p,R)$ is small in most of the range of p; it could also be viewed as roughly independent of R and p and so will be denoted as a constant c. Thus

$$\int \mathrm{d}y f(x,y) f^{-1}(y,z) \simeq \delta(x-z) + \frac{c}{2}(x-z)\delta(x-z) = \delta(x-z) \tag{3.9}$$

follows immediately. Though equations (3.4) or (3.9) could be proved approximately as above, it is still a weak point in our formalism.

From now on, we will consider Φ having a non-uniform configuration and preserve the term $\frac{1}{2}(\partial_x\Phi)^2$.

Thus the total energy of our static quantized S-G system reads

$$E = \int \mathrm{d}x\left\{\tfrac{1}{2}(\partial_x\Phi)^2 + \frac{m_{\mathrm{R}}^2}{g^2}\left(\frac{g^2}{8\pi}-1\right)(\cos g\Phi)^{8\pi/(8\pi-g^2)}\right\}. \tag{3.10}$$

The mass of a static soliton M is defined as the difference between the energy E with $\Phi = \Phi_{\mathrm{s}}$ describing a soliton configuration and that with $\Phi = 0$, i.e. (m_{R} will be simplified to m below):

$$M_{\mathrm{s}} = E_{\mathrm{s}} - E_0 = \int \mathrm{d}x\left\{\tfrac{1}{2}(\partial_x\Phi)^2 + \left(1-\frac{g^2}{8\pi}\right)\frac{m^2}{g^2}\left[1-(\cos g\Phi)^{8\pi/(8\pi-g^2)}\right]\right\}. \tag{3.11}$$

The soliton equation Φ_s should be found via the variation $\delta E/\delta\Phi = 0$ to be [9]

$$\frac{d^2\Phi}{dx^2} - \frac{m^2}{g}(\cos g\Phi)^{g^2/(8\pi-g^2)} \sin g\Phi = 0. \tag{3.12}$$

Only in the weak coupling limit $g \to 0$, can we get the well known expression for the s-G soliton

$$\Phi^{(0)}(x) = \frac{4}{g}\tan^{-1}\exp(mx). \tag{3.13}$$

Multiplying (3.12) by $d\Phi/dx$, we easily obtain after integration

$$\frac{1}{2}\left(\frac{d\Phi}{dx}\right)^2 = \left(1 - \frac{g^2}{8\pi}\right)\frac{m^2}{g^2}[1 - (\cos g\Phi)^{8\pi/(8\pi-g^2)}] \tag{3.14}$$

where the boundary conditions $\Phi = 0$, $d\Phi/dx = 0$ are used.

The comparison between (3.11) and (3.14) reveals that

$$M_s = \int_{-\infty}^{\infty} dx\left(\frac{d\Phi}{dx}\right)^2 = \int_{-\infty}^{\infty} dx\left\{2\left(1 - \frac{g^2}{8\pi}\right)\frac{m^2}{g^2}[1 - (\cos g\Phi)^{8\pi/(8\pi-g^2)}]\right\} \tag{3.15}$$

or ($g\Phi = \theta$)

$$M_s/m = \frac{1}{g^2}\left(2 - \frac{g^2}{4\pi}\right)^{1/2}\int_0^{\theta_0} d\theta[1 - (\cos\theta)^{8\pi/(8\pi-g^2)}]^{1/2}. \tag{3.16}$$

The upper bound of integration, θ_0, is fixed by the boundary condition of the soliton. Either $\theta_0 = 2\pi$ or $\theta_0 = \pi$ will be used (see section 4).

4. Analytic and numerical results

According to the expression of the GEP, equation (2.22), one finds from conditions $dV_{eff}/d\Phi = 0$ and $d^2V_{eff}/d\Phi^2 > 0$ that $\Phi = 0$ and 2π always correspond to a minimum. For the integrand of equation (3.15) not to develop an imaginary part, we take the value of coupling constant to be

$$\frac{8\pi}{8\pi - g^2} = \frac{2n+N}{2n+1} \qquad g^2 = \frac{N-1}{2n+N}8\pi \tag{4.1}$$

where $n = 0, 1, 2, \ldots, N = 1, 2, \ldots$.

Then one finds further that

$$\left.\frac{dV_{eff}}{d\Phi}\right|_{g\Phi=\pi} = 0 \tag{4.2}$$

$$\left.\frac{d^2V_{eff}}{d\Phi^2}\right|_{g\Phi=\pi} = m^2(-1)^{(2n+N)/(2n+1)}. \tag{4.3}$$

So $g\Phi = \pi$ also becomes a minimum when N = even in contrast to a maximum when N = odd. The use of two integer numbers n and N enables us to approximate any value of g^2 over the whole region $(0, 8\pi)$. Some analytical results can be obtained from (3.16) as follows.

(i) For example, in the case of $g^2=4\pi$, corresponding to $N=2$, $n=0$, we find

$$M_s=\frac{m}{g^2}\int_0^\pi d\theta \sin\theta=\frac{2m}{g^2}=\frac{m}{2\pi}. \tag{4.4}$$

Actually, in doing the computer analysis, we begin from the equation

$$x-x_0|_{x_0\to-\infty}=\int_0^\theta \frac{dx}{d\theta}d\theta$$

with $dx/d\theta$ given by equation (3.14) as a function of $\theta=g\Phi$. After setting $g^2=4\pi$, we let the computer run from $x_0\to-\infty$ as θ increases from 0. Then to our surprise, θ runs to π when $x\to\infty$; θ never runs to 2π. Alternatively, we may set a value $\theta=\theta_1$, $0<\theta_1<\pi$, at $x=0$, then the computer will run to $\theta\to 0$ at $x\to-\infty$ and $\theta\to\pi$ at $x\to+\infty$. A soliton linking $\theta=0$ and π appears clearly on the screen rather than an expected one linking $\theta=0$ and 2π. This kind of situation occurs for every value of g in (4.1) with $N=$ even.

Certainly, in the classical case, there is no soliton linking $\theta=0$ and π. We always have a classical soliton linking $\theta=0$ and 2π with mass

$$M_{cl}=\frac{8m}{g^2}. \tag{4.5}$$

So the emergence of the soliton linking $\theta=0$ to π may be a false phenomenon stemming from the approximate nature of the GEP method. But for comparison with the soliton mass at the vicinity value of g with $N=$ odd, we think it may be worth calculating the difference formally:

$$(M_{cl}-M_s)_{g^2=4\pi}=\frac{3m}{2\pi}=0.4775m. \tag{4.6}$$

(ii) An interesting case is the weak coupling limit $g^2\to 0$, corresponding to $N\to\infty$. In this case one may neglect g^2 in the exponential and get

$$M_s|_{g^2\to 0}=\lim_{g^2\to 0}\left\{\frac{m}{g^2}\left(2-\frac{g^2}{4\pi}\right)^{1/2}\int_0^{2\pi} d\theta(1-\cos\theta)^{1/2}\right\}$$

$$=\lim_{g^2\to 0}\left(\frac{8m}{g^2}-\frac{m}{2\pi}\right). \tag{4.7}$$

So

$$(M_{cl}-M_s)_{g^2\to 0}=\frac{m}{2\pi}. \tag{4.8}$$

(iii) Obviously, in the strong coupling limit $g^2\to 8\pi$, equation (3.15) gives

$$\lim_{g^2\to 8\pi} M_s\to 0 \tag{4.9}$$

and

$$(M_{cl}-M_s)_{g^2\to 8\pi}=\left.\frac{8m}{g^2}\right|_{g^2\to 8\pi}=\frac{m}{\pi}. \tag{4.10}$$

(iv) For generic value of g, we obtain the numerical result of M_s from equation (3.16) by using a computer.

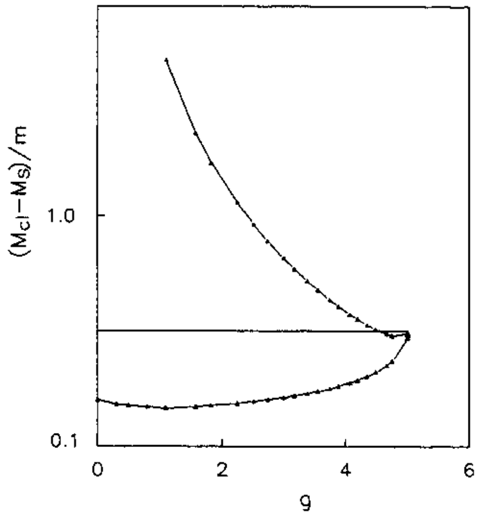

Figure 1. The difference between the classical value of a soliton mass $M_{\rm cl} = 8m/g^2$ and the quantized soliton mass M_s, in unit of m, as a function of coupling constant g. The lower curve (with full triangles) refers to soliton linking $g\Phi = 0$ and $g\Phi = 2\pi$ while the upper one (with open triangles) refers to that linking $g\Phi = 0$ and $g\Phi = \pi$. The middle solid line is for $1/\pi$, the ordinate is in logarithmic scale. For more detail, see the text.

Then two curves on the plot of $(M_{\rm cl} - M_s)/m$ versus g are obtained. They correspond to N = even or odd respectively and tend to the common limit $1/\pi$ at $g^2 \to 8\pi$ (see figure 1). Note that if the new kind of soliton does exist at the quantum level, its energy is lower than the usual one.

5. Summary and discussion

(i) We propose a non-perturbative approach for calculating the soliton mass in the quantized s-G model based on the GEP method. The crucial point lies in the observation that the effective potential (2.22) derived in the s-G model remains approximately valid for non-uniform configuration, $\Phi \neq$ constant.

(ii) A remarkable new feature of the GEP, equation (2.22), is that it develops a new minimum at $g\Phi = \pi$ when

$$g^2 = \frac{N-1}{2n+N} 8\pi \qquad N = 2, 4, \ldots \tag{5.1}$$

so a soliton connecting $g\Phi = 0$ and π appears in addition to the well known soliton connecting $g\Phi = 0$ and 2π when

$$g^2 = \frac{N-1}{2n+N} 8\pi \qquad N = 1, 3, \ldots \tag{5.2}$$

(iii) The analytical expression for soliton mass, equation (3.16), looks quite elegant with $\theta_0 = \pi$ or 2π relevant to case (5.1) or (5.2). The labour cost in computer work is much less than that in other methods, e.g. the Hartree–Fock (H-F) method in [6].

(iv) As a criterion for accuracy, we compare our result (4.7) for $g^2 \to 0$ with that of other methods. Indeed, instead of (3.16), we use equation (3.15) and then substitute the approximate solution $\Phi = \Phi^{(0)}$, equation (3.13), into it. Then we get

$$M_s|_{g^2\to 0} = 4\left(1 - \frac{g^2}{8\pi}\right)\frac{m2}{g^2}\int_{-\infty}^{\infty} \operatorname{sech}^2 mx \, \mathrm{d}x$$

$$\approx \frac{8m}{g^2} - \frac{m}{\pi} \tag{5.3}$$

which coincides precisely with that derived from the H-F method [6] or the WKB method [10]. Since equation (5.3) contains a further approximation than that in equation (3.15) which has the same accuracy as equation (3.16), so we have more confidence in result (4.7) over (5.3).

(v) Now we try to compare our GEP method with the H-F method in [6, 7]. Both methods share the common point of view that the quantum correction of the soliton mass is due to the difference between the quantum fluctuation under a soliton background and that under the uniform vacuum. The difference between these two methods lies in the different treatment of mode resolution.

In the H-F method, one works in the Heisenberg picture and expands the quantum fluctuation into modes of fictitious particles with energy square $\omega_k^2 = k^2 + m^2$ in the uniform vacuum, but into modes ω_α in the presence of a soliton. In the classical case, ω_α also has a continous spectrum like ω_k besides a lower discrete one $\omega_\alpha|_{\alpha=0} = 0$ which is called the zero mode. However, in the quantum case $\lim_{g^2\to 0} \omega_\alpha|_{\alpha=0} = 30^{-1/3} g^{2/3}$ ceases to be zero ($g \neq 0$) (see equation (4.5) of [6]).

It seems to us that a zero mode $\omega_0 = 0$ should always exist even in the quantum case because it reflects the translational invariance of the Lagrangian. So the mode resolution into ω_α (which comprises more and more discrete modes when the coupling becomes stronger) [6] does not have too many physical implications.

On the other hand, in the GEP method, we work in the Schrödinger picture and expand the quantum fluctuation f_{xy} into a Fourier integral, i.e. in the representation of plane wave e^{ipx} with continuous p spectrum, irrespective of the absence or presence of a soliton. This is not an eigenmode resolution but a mathematical trick from the beginning. Even a localized zero mode can be expanded into a Fourier integral, which does not need a particle-like explanation. For mode resolution in the GEP method, see the appendix.

(vi) There is subtlety in our formalism. As mentioned before, we take the values of $g^2/8\pi$ in (5.1) or (5.2) as special rational numbers to avoid the appearance of an imaginary part in the calculation in equation (3.16). Notice that, however, the mass parameter $\Omega = \mu$ turns out to be imaginary when $\cos g\Phi < 0$ which does occur at the central part of a soliton connecting $g\Phi = 0$ and 2π. This may imply the instability of long-wave quantum fluctuation (see (3.5) and (3.6)) around a soliton to initiate a change of ground state from $g\Phi = 0$ to 2π or vice versa. Fortunately, this complexity causes no harm to the evaluation of the soliton mass which remains real in either the (5.1) or (5.2) case.

(vii) Because the approximation of the GEP method is not under control, we tend to take a conservative attitude. Not only should the 'new' soliton linking $g\Phi = 0$ to π not be stressed, but also the discrete values of $g^2/8\pi$ may only have limited meaning. What we can claim with confidence is that for any value of $g^2/8\pi$ between 0 and 1, the GEP method can provide a mass value of a quantized soliton (linking $g\Phi = 0$ to

2π) approximately, by adjusting two integers n and N. However, there is some recent literature emphasizing the discrete coupling constants of the s-G model based on rigorous treatment [11, 12]. So as a bold conjecture, we might raise the following question. Do the discrete values of $g^2/8\pi$ discussed in this paper have some relevance to the intrinsic symmetry of s-G model? Further investigation is needed.

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Appendix. Zero mode, the kinetic energy and excitation of a soliton

Notice first that we have ignored a kinetic energy term $\frac{1}{2}\mathscr{P}_x^2$ in discussing the mass of a soliton (see equation (2.8)). If replace $\mathscr{P}_x = \langle\Psi|\Pi_{\Phi_x}|\Psi\rangle$ by $\partial\Phi/\partial t$, we can write the effective action as

$$S_{\mathrm{eff}}[\Phi] = \int \mathrm{d}t\, \mathrm{d}x\{\tfrac{1}{2}(\partial_t\Phi)^2 - \tfrac{1}{2}(\partial_x\Phi)^2 - V_{\mathrm{eff}}(\Phi)\}. \tag{A.1}$$

with $V_{\mathrm{eff}}(\Phi)$ given by (2.22).

The variation $\delta S_{\mathrm{eff}} = 0$ yields the motion equation:

$$\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} + V'_{\mathrm{eff}}(\Phi) = 0. \tag{A.2}$$

Hence we see that a moving quantized soliton is boosted from a static one

$$\Phi_s(x) \to \Phi_s(\xi) = \Phi_s\left(\frac{x - vt}{\sqrt{1 - v^2}}\right) \tag{A.3}$$

with $\Phi_s(x)$ satisfying (A.2) without the first term, i.e. equation (3.12).

Let us evaluate the energy of a moving soliton via the following effective Hamiltonian:

$$H_{\mathrm{eff}}[\Phi] = \int \mathrm{d}x\{\tfrac{1}{2}\mathscr{P}_x^2 + \tfrac{1}{2}(\partial_x\Phi)^2 + V_{\mathrm{eff}}(\Phi) - V_{\mathrm{eff}}(0)\}. \tag{A.4}$$

Substituting (A.3) into (A.4) and using (3.11), (3.14) and (3.15), one finds

$$\begin{aligned} E[\Phi_s(\xi)] &= \int \mathrm{d}\xi\, \sqrt{1 - v^2}\left\{\frac{1}{2}\,\frac{1 + v^2}{1 - v^2}[\Phi'_s(\xi)]^2 + V_{\mathrm{eff}}(\Phi_s) - V_{\mathrm{eff}}(0)\right\} \\ &= \int \mathrm{d}\xi\, \sqrt{1 - v^2}\left\{\frac{1}{1 - v^2}[\Phi'_s(\xi)]^2\right\} = \frac{M_s}{\sqrt{1 - v^2}} \end{aligned} \tag{A.5}$$

as expected.

Next, we consider how the effective potential changes when the configuration $\Phi(x)$ deviates a little from a static soliton $\Phi_s(x)$:

$$E[\Phi]=E[\Phi_s]+\int \mathrm{d}x\frac{1}{2}\left\{\eta(x)\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V''(\Phi_s)\right]\eta(x)+\ldots\right\} \tag{A.6}$$

with

$$\eta(x)=\Phi(x)-\Phi_s(x)$$

$$V''(\Phi_s)=\left.\frac{\mathrm{d}^2V_{\mathrm{eff}}}{\mathrm{d}\Phi^2}\right|_{\Phi=\Phi_s}.$$

Similar to the discussion in [1], we have the eigenvalue problem:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V''_{\mathrm{eff}}(\Phi_s)\right]\eta_i(x)=\omega_i^2\eta_i(x) \tag{A.7}$$

where $\eta_i(x)$ are the orthonormal 'normal modes' of fluctuations around $\Phi_s(x)$. So the time-dependent fluctuation reads

$$\eta(x,t)=\Phi(x,t)-\Phi_s(x)=\sum_{i=0}^{\infty}C_i(t)\eta_i(x). \tag{A.8}$$

Note that $i=0$ refers to the zero mode $\eta_0(x)$ satisfying

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V''_{\mathrm{eff}}(\Phi_s)\right]\eta_0(x)=0. \tag{A.9}$$

It can be seen from performing $\mathrm{d}/\mathrm{d}x$ on the motion equation of $\Phi_s(x)$ that $\eta_0(x)=(\mathrm{d}/\mathrm{d}x)\Phi_s(x)$ with $\omega_0=0$ as expected. Finally, the total energy of an excited moving quantized soliton should be

$$E(v,\{n_i\})=\frac{M_s}{\sqrt{1-v^2}}+\hbar\sum_i\eta_i\omega_i+\text{corrections}. \tag{A.10}$$

We wish to stress once more that:

(i) There are no zero point energies of vacuum quantum fluctuation, $\frac{1}{2}\Sigma_i\,\hbar\omega_i$, in the second term. They have already been absorbed into the mass of the soliton, M_s.

(ii) The zero mode $\omega_0=0$ contributes nothing to the second term. The excitation energy related to translational invariance has been ascribed to the kinetic energy of the soliton in the first term.

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